# Students' Mental Models and Schema Activation during Geometric Problem Solving

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*Recent investigations of mathematical problem solving have focused on issues that affect students' ability at accessing and making flexible use of previously learnt knowledge. We report here the first phase of a study that takes up this issue by examining potential links that might exist between mental models constructed by students, the organisational quality of students' prior geometric knowledge and the use of that knowledge during problem solving. The results suggest that the quality of geometric knowledge that students construct could have a powerful effect on their mental models and subsequent use of that knowledge.*

## Background

Despite the extensive research on problem solving, students continue to show less than satisfactory levels of performance in situations where they are required to apply previously learnt mathematical knowledge to the solution of new problems (Board of Senior Secondary Studies, 1994). This can be largely attributed to the poor quality of the students' mathematical knowledge base (Resnick and Ford, 1981). It has been argued that a well-organised knowledge base not only facilitates access of appropriate information, but also determines how this knowledge is deployed in the search for a problem solution (Prawat,1989; Alexander & Judy, 1988; Lawson & Chinnappan, 1994). Research on mathematical problem solving needs to give greater attention to the nature of the knowledge structures students bring to a problem situation, the extent to which they utilize these during the solution process, and the effectiveness with which they do so. These issues formed the focus of a study of students' geometric problem solving reported here.

Research in the area of cognitive psychology and human problem solving (Mayer, 1975; Kintsch and Greeno, 1985, Halford, 1993) has generated useful paradigms for addressing the nature and role of knowledge in students' problem solving. In particular, the notions of schema and mental models serve as powerful constructs here. The notion of schema has been variously defined in the literature, however for our present purposes we adopt Rumelhart and Ortony's (1977) view that schemas are data structures for organising information in memory. Knowledge structures in the form of schemas guide both information acceptance, retrieval, and use. From this perspective, when students acquire mathematical concepts, principles, and procedures they organise these into schemas which provide the knowledge base for further mathematical activity. As students reflect upon and experiment with what they have learnt, they modify their mathematical schemas through a process of construction and reconstruction. It has been argued that the complexity and sophistication of these schemas have a major influence on problem categorisation (Sweller, 1989), and hence, on whether and how mathematical knowledge is utilised in the process of problem representation or categorisation (Glaser, 1984). A useful construct for addressing students' application of knowledge during problem representation and solution is that of mental models.

The term, mental models has been used extensively in the psychological literature to describe the cognitive representations individuals construct in various learning situations (e.g., English, forthcoming; English & Halford, 1995; Halford, 1993; Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991; Rouse & Morris, 1986). The notion we adopt here is that of Halford (1993), namely, mental models are representations that are active while solving a particular problem and that provide the workspace for inference and mental operations. These cognitive representations are considered the workspace of thinking and understanding and must have a high degree of correspondence to the environment that they represent. The significance of mental models for mathematics learning is their relational structure. The mental models we try to help our students construct are those in which the essential relations and principles of a mathematical domain are represented (English & Halford, 1995).

We applied these constructs to an analysis of students' knowledge access and use as they solved the plane geometry problem of the present study. Our aim was to identify the geometric schemas students bring to a problem-solving task, the frequency with which these are activated, and the nature of the mental models students utilize and/or construct during the course of problem solution. Of particular interest was how these components differ between low and high achieving students. We hypothesized that the high achievers would show superior performance because they would possess more sophisticated geometric schemas, would activate these more readily, and would utilize mental models which comprise a structural understanding of the task domain (cf. findings of English, forthcoming).

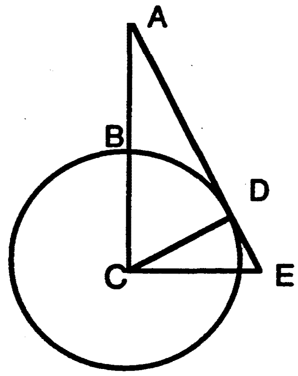
## Method

### Participants

Thirty students from five year 10 mathematics classes in a middle-class private suburban high school participated in the study. The school has a reputation for high academic standards and innovative teaching practices. Fifteen of the students were classified as high achievers and 15 as low achievers, based on recent class tests.

### Materials and Procedure

The students were individually administered the plane geometry problem shown in Fig. 1. The problem was presented on an index card and students were encouraged to talk as much as possible as they attempted to solve it. Each student's response was video recorded.




AE is a tangent to the circle, centre C. AC is perpendicular to CE, and the angle DCE has a measure of 30 degrees The radius of the circle is equal to 5cm. Find AB.

Fig. 1: Problem presented to students

The problem consists of three commonly encountered geometric forms: a circle, tangent, and triangles. However, the problem is made more complex by having these forms integrated in a manner which demands that the solver recognizes a component as serving more than one function. For example, side AE needs to be identified as a) a straight line, b) a tangent to the circle, and c) the hypotenuse of the right-angled triangle ACE. This recognition constitutes an important prerequisite phase before appropriate theorems and formulae can be activated and new information generated.

In order to develop a checklist of geometric schemas that could be utilised in solving the above problem, we asked the following participants to solve it: a professional mathematician, two senior teachers of high school mathematics and two high school students from another school. On the basis of their solution transcripts and responses to interview questions, we identified a total of 17 geometric schemas.

## Results

One of the investigators and an experienced research assistant, both of whom had considerable experience teaching geometry at the high school level, independently examined the transcripts of our Year 10 students' solution attempts for the type and frequency of use of geometric schemas.

A schema, such as 'right-angled triangle,' was considered to have been activated if a student explicitly mentioned it or used a trig ratio within that triangle. There was a high degree of agreement (90%) between the codings produced by the investigator and the research assistant. A particular schema was considered to have been activated more than once if that schema was used on a second occasion to generate further new information or in the exploration of an alternative path to the solution.

The results of this analysis showed that the high-achieving students not only activated a greater variety of geometric schemas than the low achievers, but they also activated more than four times the number of schemas of their peers. The tangent-radius theorem and perpendicularity were the most actively used schemas by both groups. The low achievers did not activate seven of the 17 schemas that could have been accessed, in contrast to the high achievers who activated all but three of the schemas. This suggests that the low achievers either did not possess this information or in fact, did possess it but could not see its relevance to the present problem.

These findings however, do not provide much information regarding <u>how</u> the schemas were accessed. In particular, the data do not throw much light on the possible links between these schemas and the information embedded in the problem, and the effect such connections might have on schema access. To explore these issues, we re-examined students' solution attempts with a focus on their approaches to solution. As a basis for categorising these approaches, we drew upon the responses of the reference group mentioned earlier (i.e., the professional mathematician, senior teachers and two students). Their approaches to the problem displayed an identifiable solution path and were as follows:

1. <u>Trigonometry</u>

This approach involves the exclusive use of trigonometric ratios, that is, tangent, cosine and sine, during the solution search. One or more of these ratios are used in a right-angled triangle (s) to generate the length of AC. The required length of AB is then found by subtracting BC from AC.

2. <u>Equilateral/isosceles triangles</u>

This approach involves the construction of the line joining points B and D, a move that helps create an equilateral triangle (BCD), and an isosceles triangle (ABD). By using a series of deductive reasoning processes, the solver is able to infer that BD has a length of 5cm, and that AB is equal in length to BD.

3 <u>Trigonometry/Pythagoras' theorem</u>

This approach follows the moves described in the first approach except that, instead of using one of the trig ratios, Pythagoras' theorem is applied to triangle ADC to generate the length of AC. The length of BC is subsequently subtracted from AC to work out the length of AB.

4 Similar triangles/trigonometry

This method incorporates trigonometric ratios and the Pythagoras rule in the solution attempt. In this approach the solver works out the lengths of one or more sides of the right-angled triangles CDE, ACD and ACE. The solver then recognises the similarity between these triangles. This move is used to set up an equation showing ratios between corresponding sides of the triangles. The equation is then solved to find the length of AD. The length of AC is then determined by the application of trig ratios to triangle ADC. Finally, AB is found by the same moves described in the first approach.

Any solution attempt which did not have a definable path was classified as non-discernible. The numbers of students in each achievement group who demonstrated each approach appear in Table 1.

**Table 1:** Frequency of Use of Each Solution Approach by Students in Each Achievement Group

| Solution Approach | Low-Achievers | High-Achievers |
|---|---|---|
| Trigonometry | 5 | 12 |
| Equilateral/Isosceles Triangles | 0 | 0 |
| Trigonometry/Pythagoras | 1 | 1 |
| Similar Triangles/Trigonometry | 0 | 2 |
| Not discernable | 9 | 0 |
| Total | 15 | 15 |

As can be discerned from Table 1, nine of the low-achieving students did not use a definable solution approach, in contrast to the high achievers who all did so. It is not surprising then, that none of the low-achieving students produced the correct solution, whereas 11 of the high-achieving students were able to do so. These findings suggest that the low-achieving students adopt a problem-solving approach in which available geometric schemas are applied at random, with little focus on the problem goal. For example, a student might recognise the line AE as a tangent, and the line CD as a radius. On the basis of previous experience with the tangent and radius, the student generates the angle CDE as a right angle. Although this constitutes a correct move, it is of little assistance if that angle is not used to find the length of AB. In contrast, students in the high-achieving group make use of these two sets of information and an appropriate trig ratio to find the length of AC which in turn enables them to find the length of AB. This latter move involving trigonometric approach that is adopted by the high achievers shows that they are able to establish the required connections between significant components of the problem.

The trigonometry approach was the most frequently used by both groups of students, with the high achievers favouring this over all other approaches. The trigonometry/Pythagoras, similar triangles/trigonometry, and equilateral/isosceles approaches were the least favoured by both groups of students. The dominance of the trigonometry approach over all others suggests that the mental models students utilized during problem solution largely comprised trigonometric knowledge and associated information such as the properties of right-angled triangles.

The identification of students' geometric schemas and solution approaches provided information about the connections that students appeared to make between features of the problem and their existing knowledge of trigonometry and geometry. In order to examine these connections further, we analysed each of the students' solution transcripts with a view to determining the extent of these connections and their relationship to goal attainment. This information would provide some insight into the nature of the mental models students constructed during the course of problem solution.

We analysed the solution approaches adopted by a high achieving student (Michael) and a low achieving student (David) respectively. Both students appeared to have activated a number of schemas such as the tangent-radius theorem, perpendicularity, complementarity, and the properties of the right-angled triangle. Michael however, was able to infer that CB is equal in length to CD, that is, the radius of the circle also formed part of the length of one side of the right-angled triangle. He then linked this information to the cosine ratio to work out the length of the required segment, AB. David generated the magnitude of angle ACD but did not exploit this information in any purposeful way. He then attempted to use the sine ratio on triangle ACD, a move that did not help, and ultimately lead him to abandon the solution attempt. It is possible that David could have restarted his solution attempt, however he did not do so.

It could be inferred that the mental models Michael employed during problem solution drew upon a greater range of geometric schemas than those of David. More importantly, these models generated an extended chain of actions which ultimately lead to the solution of the problem. In contrast, the models constructed by the low achieving student generated information that had the potential to arrive at the solution but was not exploited appropriately and hence, the goal was not achieved.

## Discussion

In this study we have attempted to analyse the geometric schemas activated by a group of high and low achieving students as they worked through a geometry problem. The results of the study suggest that the problem-solving approaches used by the high achieving students involved the activation of more varied and complex schemas than those generated by the low achieving students. In characterising these solution approaches, we focused on the type and frequency of geometric schemas that were activated and used by the two groups.

We hypothesised that during the solution attempt, students constructed mental models of the problem by identifying its salient features and aligning these with components of their existing knowledge (Gentner,1983). The quality of the mental models constructed by the students in the two groups differed significantly, however. The high achieving students attended to the structural features of the problem and were thus able to form meaningful, integrated mental representations. The low achievers, in contrast, tended to focus on the superficial aspects and hence did not see the connections among the important features of the problem. This difference in problem perception is a well documented finding of studies on novice/expert problem solvers (Chi, Glaser & Rees, 1982; Chiesi, Spilich & Voss, 1979; De Jong & Ferguson-Hessler,1986). Both Michael and David were able to identify the right-angled triangle ACD. However, Michael could notice the relevance of this triangle in determining the length of the segment AC, which he subsequently used to find AB. David, on the other hand, could not identify the link between these two pieces of geometric information. It could be argued that Michael's availability of a richer network of geometric schemas assisted in his construction of more sophisticated mental models that comprised the important structural elements necessary for problem solution. Furthermore, Michael appeared more active in constructing these models and more purposeful in applying them during the solution process.

In sum, the results of this study shows that the accessing of mathematical knowledge during problem-solving involves complex processing of information given in the problem and prior knowledge. It appears that the quality of organisation of prior geometric knowledge plays an important role in facilitating use of that knowledge, thus providing support for claims made by Prawat (1989).The data we have generated, however, provide only a partial picture about possible interaction between the state of organisation of available geometric knowledge and the accessing of this knowledge during problem solving (Alexander and Judy, 1988). We have provided explanations about knowledge access and use from the framework of mental models. More work is needed in this area of mental models and the accessing of available knowledge in the domain of geometry.

## Implications for geometry learning and teaching

The results of the present study have direct implications for the way geometry is taught and assessed in the classroom. The teaching of geometry needs to foster students' construction of rich and elaborate schemas, which will enable them to solve a range of novel and routine geometric problems. We need to provide students with opportunities to experiment with core geometrical concepts. For example, in addition to introducing the sine ratio with reference to a right-angled triangle, teachers should encourage students to change the orientation of the right-angled triangle and explore the use of the sine ratio. More opportunities should also be provided for students to identify a particular form which is embedded in a composite figure. We need to encourage students to look for novel connections among the concepts they meet. Assessment procedures should value such connections.

An effective way to foster students' construction of more complex geometric structures is through a mentoring process in which the teacher guides and models the use geometric knowledge in a variety of situations. Classroom practices, should develop these activities and challenge students to move to more novel areas in which the use of geometric knowledge could be explored meaningfully. This instructional strategy is consistent with Vygotsky's (1978) philosophy, namely, that we need to help students progress towards their zone of proximal development in the geometric domain.

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